Words

By a *word*, I mean a *finite or infinite sequence* of symbols (*letters*) taken from a non-empty finite set $A$ (*alphabet*).

**Examples:**

- 001

- $(001)_{\infty} = 001001001001001001001001001001\ldots$

- 110011110001101110111001101110010111111101\ldots

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The study of combinatorial properties of words, known as \textbf{combinatorics on words} (MSC: 68R15), has connections to many modern, as well as classical, fields of mathematics with applications in areas ranging from \textbf{theoretical computer science} (from the algorithmic point of view) to \textbf{molecular biology} (DNA sequences).
Combinatorics on Words

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Many easily formulated problems are difficult to solve ... mainly because of the limited availability of mathematical tools to deal with non-commutative structures compared to commutative ones.
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- Recent developments in combinatorics on words have culminated in the publication of three books by a collection of authors, under the pseudonym of M. Lothaire . . .
Combinatorics on Words . . .

Lothaire Books

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Combinatorics on words has now become a very active and challenging field in its own right.
Combinatorics on Words ...

**Background**

Combinatorics on Words

\[ \text{Number Theory} \]

\[ \text{Discrete Geometry} \]

\[ \text{Probability Theory} \]

\[ \text{Discrete Dynamical Systems} \]

\[ \text{Biology} \]

\[ \text{Theoretical Computer Science} \]

\[ \text{Logic} \]

\[ \text{Algebra} \]

\[ \text{Free Groups, Semigroups} \]
\[ \text{Matrices} \]
\[ \text{Representations} \]
\[ \text{Burnside Problems} \]
Basic Definitions: Rotations

Given a word $w = x_1 x_2 \cdots x_n$ (with $x_i$ letters), the first rotation of $w$ is $R(w) = x_2 \cdots x_n x_1$. 
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More precisely, any word $w$ can be uniquely expressed in the form

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In particular, we note that any primitive word $w$ has exactly $|w|$ distinct rotations.
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Equivalently, $x$ and $y$ are conjugate if and only if there exists a word $z$ such that $xz = zy$.
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- This is an equivalence relation on \( A^* \) since \( x \) is conjugate to \( y \) if and only if \( y \) can be obtained by a cyclic permutation (rotation) of the letters of \( x \).
Basic Definitions: Necklaces

The set of all conjugates (rotations) of a given word is called its **conjugacy class**.
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![Diagram of necklaces](image)

**Note:**

A necklace of length $n$ over a $k$-letter alphabet can be thought of as $n$ circularly connected beads of up to $k$ different colours.
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Each aperiodic (primitive) necklace can be uniquely represented by a so-called **Lyndon word** ...
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Then we can totally order the semigroup $\mathcal{A}^+$ by the \textit{lexicographical order} $\preceq$ (which is the usual \textit{alphabetical order} in a dictionary) induced by the total order $\prec$ on $\mathcal{A}$. 
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A finite word $w \in \mathcal{A}^+$ is a Lyndon word if $w$ is strictly smaller in lexicographical order than all of its non-trivial rotations for the given total order $\prec$ on $\mathcal{A}$.
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Lyndon words are named after mathematician Roger Lyndon, who introduced them in 1954 under the name of standard lexicographic sequences.
Suppose the alphabet $\mathcal{A}$ is totally ordered by the relation $\prec$.

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A finite word $w \in \mathcal{A}^+$ is a **Lyndon word** if $w$ is strictly smaller in lexicographical order than all of its non-trivial rotations for the given total order $\prec$ on $\mathcal{A}$.

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It follows that $w \in \mathcal{A}^+$ is a Lyndon word iff $w \in \mathcal{A}$ or $w \prec v$ for all proper suffixes $v$ of $w$. 
Examples

Example 1: $w = aabac$ with alphabet $\mathcal{A} = \{a, b, c\}$
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Example 2

For the 2-letter alphabet \( \{a, b\} \) with \( a \prec b \), the Lyndon words up to length five are as follows (sorted lexicographically for each length):

\[ a, b, ab, aab, abb, aabb, aabbb, aaabb, aabab, aabbb, abbbb, \ldots \]
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\text{Length} & \quad \text{Words} \\
2 & \quad a, b, ab \\
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• Lyndon words also have applications to semigroups, pattern matching, and representation theory of certain algebras (cf. recent work of Ram et al.).
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I’ll return to some of the aforementioned applications along the way . . .
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From now on, we consider words over a totally ordered finite alphabet $\mathcal{A}$ consisting of at least two distinct letters, unless stated otherwise.
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where $d = |v|$ and the number of distinct conjugates of $w$ is exactly $d$.

Hence, the total number of different words of length $n$ on $\mathcal{A}$ is

$$k^n = \sum_{d|n} d \cdot N_k(d)$$

where the sum is over all positive divisors of $n$. 
Now, by the well-known Möbius inversion formula, we have

\[
N_k(n) = \frac{1}{n} \sum_{d | n} \mu(d) \cdot k^{n/d}
\]

where \(\mu\) is the Möbius function defined on \(\mathbb{N}^+\) as follows:

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\mu(n) = \begin{cases} 
1 & \text{if } n = 1, \\
(-1)^i & \text{if } n = p_1 \cdots p_i \text{ where the } p_i \text{ are distinct primes,} \\
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This “necklace-counting formula” (often called Witt’s formula) was proved by E. Witt in 1937 in connection with the theorem on free Lie algebras now called the Poincaré–Birkhoff–Witt Theorem.
By Witt’s formula, the numbers of binary Lyndon words of each length, starting with length 0 (empty word), form the integer sequence

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At the core of Duval’s efficient algorithm is the following important factorisation theorem ...
Factorisation of Lyndon words

**Theorem (Lyndon)**

A word $w \in A^+$ is a Lyndon word if and only if $w \in A$ or there exists two Lyndon words $u$ and $v$ such that $w = uv$ and $u \prec v$. 
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In general, this factorisation is not unique since, for example, $w = aabac$ has two such factorisations:

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**Theorem (Chen-Fox-Lyndon 1958)**

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**Theorem (Chen-Fox-Lyndon 1958)**

If \( w = uv \) is a Lyndon word with \( v \) its lexicographically smallest proper suffix, then \( u \) and \( v \) are also Lyndon words and \( u \prec v \).

So the standard factorisation of a Lyndon word \( w = uv \) is obtained by choosing \( v \) to be the lexicographically least proper suffix of \( w \).
Factorisation of Lyndon words

**Theorem (Lyndon)**

A word $w \in A^+$ is a Lyndon word if and only if $w \in A$ or there exists two Lyndon words $u$ and $v$ such that $w = uv$ and $u \prec v$.

In general, this factorisation **is not unique** since, for example, $w = aabac$ has two such factorisations:

$$w = (a)(abac) \quad \text{and} \quad w = (aab)(ac).$$

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If $w = uv$ is a Lyndon word with $v$ its lexicographically smallest proper suffix, then $u$ and $v$ are also Lyndon words and $u \prec v$.

So the **standard factorisation** of a Lyndon word $w = uv$ is obtained by choosing $v$ to be the lexicographically least proper suffix of $w$, which also happens to be the longest proper suffix of $w$ that is Lyndon.
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Example: \( w = aabac \) has standard factorisation \( w = (a)(abac) \).
In algebraic settings, Lyndon words give rise to commutators using standard factorisation iteratively.
An Application in Algebra

- In algebraic settings, Lyndon words give rise to **commutators** using standard factorisation iteratively.

- For example, the Lyndon word $aababb$ with standard factorisation $(a)(ababb)$ gives rise to the commutator $[a, [[a, b], [[a, b], b]]]$. 
An Application in Algebra

- In algebraic settings, Lyndon words give rise to **commutators** using **standard factorisation** iteratively.

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- These commutators can be viewed either as elements of the **free group** with $[x, y] = xyx^{-1}y^{-1}$, or as elements of the **free Lie algebra** with $[x, y] = xy - yx$. 
In algebraic settings, Lyndon words give rise to **commutators** using standard factorisation iteratively.

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These commutators can be viewed either as elements of the **free group** with $[x, y] = xyx^{-1}y^{-1}$, or as elements of the **free Lie algebra** with $[x, y] = xy - yx$.

In either case, Lyndon words give rise to a basis of some algebra.
An Application in Algebra

- In algebraic settings, Lyndon words give rise to **commutators** using **standard factorisation** iteratively.

- For example, the Lyndon word \(aababb\) with standard factorisation \((a)(ababb)\) gives rise to the commutator \([a, [[a, b], [[a, b], b]]]\).

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Generation of Lyndon Words

**Duval’s Algorithm:** Generates the Lyndon words over $A$ of length at most $n$ ($n \geq 2$).
Generation of Lyndon Words

**Duval’s Algorithm**: Generates the Lyndon words over $\mathcal{A}$ of length at most $n$ ($n \geq 2$).

If $w$ is one of the words in the list of Lyndon words up to length $n$ (not equal to $\text{max}(\mathcal{A})$), then the next Lyndon word after $w$ can be found by the following steps:
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1. Repeat the letters from $w$ to form a new word $x$ of length exactly $n$, where the $i$-th letter of $x$ is the same as the letter at position $i \pmod{|w|}$ in $w$. 
Generation of Lyndon Words

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2. If the last letter of \( x \) is \( \max(\mathcal{A}) \) for the given order on \( \mathcal{A} \), remove it, producing a shorter word, and take this to be the new \( x \).
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Generation of Lyndon Words

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**Generation of Lyndon Words**

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**Note**: Since, in general, the factorisation of a Lyndon word as a product $uv$ of two Lyndon words $u, v$ with $u \prec v$ is not unique, Duval’s algorithm may produce the same Lyndon word more than once.
Duval’s Algorithm in Action

Let’s use the algorithm to generate all the Lyndon words up to length 4 over the alphabet \{a, b\} with \(a < b\).
Duval’s Algorithm in Action

Let’s use the algorithm to generate all the Lyndon words up to length 4 over the alphabet \( \{a, b\} \) with \( a \prec b \).

We begin with the Lyndon words of length 1: \( a \) and \( b \).
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We begin with the Lyndon words of length 1: \(a\) and \(b\). List: \(\mathcal{L} = \{a, b, \ldots\}\)

Lyndon words of length at most 2:

\[a\]
Duval’s Algorithm in Action

Let’s use the algorithm to generate all the Lyndon words up to length 4 over the alphabet \(\{a, b\}\) with \(a \prec b\).

We begin with the Lyndon words of length 1: \(a\) and \(b\). List: \(L = \{a, b, \ldots\}\)

Lyndon words of length at most 2:

\[
a \rightarrow aa
\]
Duval’s Algorithm in Action

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We begin with the Lyndon words of length 1: \( a \) and \( b \). List: \( \mathcal{L} = \{a, b, \ldots\} \)

Lyndon words of length at most 2:

\[
\begin{align*}
    a & \rightarrow aa & \rightarrow ab
\end{align*}
\]
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Lyndon words of length at most 2:

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  b & \rightarrow ab
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We begin with the Lyndon words of length 1: \( a \) and \( b \). List: \( \mathcal{L} = \{a, b, \ldots\} \)

Lyndon words of length at most 2:

\[
\begin{align*}
    \boxed{a} & \rightarrow aa & \rightarrow \boxed{ab} \\
    \boxed{b} & \rightarrow bb
\end{align*}
\]
Duval’s Algorithm in Action

Let’s use the algorithm to generate all the Lyndon words up to length 4 over the alphabet \( \{a, b\} \) with \( a \prec b \).

We begin with the Lyndon words of length 1: \( a \) and \( b \). List: \( \mathcal{L} = \{a, b, \ldots\} \)

Lyndon words of length at most 2:

\[
\begin{array}{c}
\text{List: } L = \{a, b, \ldots\} \\
\end{array}
\]

Let’s use the algorithm to generate all the Lyndon words up to length 4 over the alphabet \( \{a, b\} \) with \( a \prec b \).

We begin with the Lyndon words of length 1: \( a \) and \( b \). List: \( \mathcal{L} = \{a, b, \ldots\} \)

Lyndon words of length at most 2:

\[
\begin{array}{c}
a \longrightarrow aa \longrightarrow ab \\
\end{array}
\]

\[
\begin{array}{c}
b \longrightarrow bb \longrightarrow b \\
\end{array}
\]
Duval’s Algorithm in Action

Let’s use the algorithm to generate all the Lyndon words up to length 4 over the alphabet \( \{a, b\} \) with \( a \prec b \).

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Lyndon words of length at most 2:

\[
\begin{align*}
[a] & \rightarrow aa & & \rightarrow [ab] \\
[b] & \rightarrow bb & & \rightarrow b
\end{align*}
\]
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Lyndon words of length at most 2:

\[
\begin{align*}
\text{ } a & \rightarrow aa \rightarrow ab \\
\text{ } b & \rightarrow bb \rightarrow b \rightarrow \varepsilon
\end{align*}
\]
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We begin with the Lyndon words of length 1: \( a \) and \( b \). List: \( L = \{ a, b, \ldots \} \)

Lyndon words of length at most 2:

\[
\begin{align*}
a & \rightarrow a a \rightarrow \begin{array}{c}a \ b\end{array} \\
b & \rightarrow b b \rightarrow b \rightarrow \varepsilon
\end{align*}
\]
Duval’s Algorithm in Action

Let’s use the algorithm to generate all the Lyndon words up to length 4 over the alphabet \( \{a, b\} \) with \( a < b \).

We begin with the Lyndon words of length 1: \( a \) and \( b \). List: \( \mathcal{L} = \{a, b, \ldots\} \)

Lyndon words of length at most 2:

- \( a \rightarrow aa \rightarrow ab \)
- \( b \rightarrow bb \rightarrow b \rightarrow \varepsilon \)

Updated List: \( \mathcal{L} = \{a, b, ab, \ldots\} \)
Duval’s Algorithm in Action

Let’s use the algorithm to generate all the Lyndon words up to length 4 over the alphabet \{a, b\} with \(a \prec b\).

We begin with the **Lyndon words of length 1**: \(a\) and \(b\). List: \(\mathcal{L} = \{a, b, \ldots\}\)

**Lyndon words of length at most 2:**

\[
\begin{align*}
\begin{array}{c}
a \\
\end{array} & \rightarrow \begin{array}{c} aa \\
\end{array} \rightarrow \begin{array}{c} ab \\
\end{array} \\
\begin{array}{c}
b \\
\end{array} & \rightarrow \begin{array}{c} bb \\
\end{array} \rightarrow b \rightarrow \varepsilon
\end{align*}
\]

Updated List: \(\mathcal{L} = \{a, b, ab, \ldots\}\)

**Lyndon words of length at most 3:**
Duval’s Algorithm in Action

Let’s use the algorithm to generate all the Lyndon words up to length 4 over the alphabet \( \{a, b\} \) with \( a \prec b \).

We begin with the Lyndon words of length 1: \( a \) and \( b \). List: \( \mathcal{L} = \{a, b, \ldots\} \)

Lyndon words of length at most 2:

- \( a \rightarrow aa \rightarrow ab \)
- \( b \rightarrow bb \rightarrow b \rightarrow \varepsilon \)

Updated List: \( \mathcal{L} = \{a, b, ab, \ldots\} \)

Lyndon words of length at most 3:

- \( a \)
Duval’s Algorithm in Action

Let’s use the algorithm to generate all the Lyndon words up to length 4 over the alphabet \(\{a, b\}\) with \(a \prec b\).

We begin with the Lyndon words of length 1: \(a\) and \(b\). List: \(\mathcal{L} = \{a, b, \ldots\}\)

Lyndon words of length at most 2:

- \(a \rightarrow aa \rightarrow ab\)
- \(b \rightarrow bb \rightarrow b \rightarrow \varepsilon\)

Updated List: \(\mathcal{L} = \{a, b, ab, \ldots\}\)

Lyndon words of length at most 3:

- \(a \rightarrow aaa\)
Duval’s Algorithm in Action

Let’s use the algorithm to generate all the Lyndon words up to length 4 over the alphabet \( \{a, b\} \) with \( a \prec b \).

We begin with the Lyndon words of length 1: \( a \) and \( b \). List: \( \mathcal{L} = \{a, b, \ldots\} \)

Lyndon words of length at most 2:

\[
\begin{align*}
\text{List: } \mathcal{L} &= \{a, b, \ldots\} \\
\text{Updated List: } \mathcal{L} &= \{a, b, ab, \ldots\}
\end{align*}
\]

Lyndon words of length at most 3:

\[
\begin{align*}
\text{List: } \mathcal{L} &= \{a, b, ab, \ldots\}
\end{align*}
\]
Duval’s Algorithm in Action

Let’s use the algorithm to generate all the Lyndon words up to length 4 over the alphabet \( \{a, b\} \) with \( a \prec b \).

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Lyndon words of length at most 2:

\[
\begin{align*}
  a & \rightarrow aa \\
  a & \rightarrow ab \\
  b & \rightarrow bb \\
  b & \rightarrow b \\
  \varepsilon &
\end{align*}
\]

Updated List: \( \mathcal{L} = \{a, b, ab, \ldots\} \)

Lyndon words of length at most 3:

\[
\begin{align*}
  a & \rightarrow aaa \\
  a & \rightarrow aab \\
  ab & 
\end{align*}
\]
Duval’s Algorithm in Action

Let’s use the algorithm to generate all the Lyndon words up to length 4 over the alphabet \{a, b\} with \(a \prec b\).

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Lyndon words of length at most 2:

\[
\begin{align*}
  a & \rightarrow aa \\
  b & \rightarrow bb \\
  b & \rightarrow b \\
\end{align*}
\]

Updated List: \(\mathcal{L} = \{a, b, ab, \ldots\}\)

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Lyndon words of length at most 2:

- \( a \rightarrow aa \rightarrow ab \)
- \( b \rightarrow bb \rightarrow b \rightarrow \varepsilon \)

Updated List: \( \mathcal{L} = \{a, b, ab, \ldots\} \)

Lyndon words of length at most 3:

- \( a \rightarrow aaa \rightarrow aab \)
- \( ab \rightarrow aba \rightarrow abb \)
Duval’s Algorithm in Action

Let’s use the algorithm to generate all the Lyndon words up to length 4 over the alphabet \( \{a, b\} \) with \( a < b \).

We begin with the Lyndon words of length 1: \( a \) and \( b \). List: \( \mathcal{L} = \{a, b, \ldots\} \)

Lyndon words of length at most 2:

\[
\begin{align*}
  & a \rightarrow aa \rightarrow ab \\
  & b \rightarrow bb \rightarrow b \rightarrow \varepsilon
\end{align*}
\]

Updated List: \( \mathcal{L} = \{a, b, ab, \ldots\} \)

Lyndon words of length at most 3:

\[
\begin{align*}
  & a \rightarrow aaa \rightarrow aab \\
  & ab \rightarrow aba \rightarrow abb
\end{align*}
\]

Updated List: \( \mathcal{L} = \{a, b, ab, aab, abb, \ldots\} \)
Duval’s Algorithm in Action . . .

Lyndon words of length at most 4:
Duval’s Algorithm in Action . . .

Lyndon words of length at most 4:

\[ a \]
Duval’s Algorithm in Action ... 

Lyndon words of length at most 4:

\[ a \rightarrow aaaa \]
Duval’s Algorithm in Action . . .

Lyndon words of length at most 4:

\[ a \rightarrow aaaa \rightarrow aaab \]
Duval’s Algorithm in Action . . .

Lyndon words of length at most 4:

\[ a \rightarrow \text{aaaa} \rightarrow \text{aaab} \]

\[ ab \]
Duval’s Algorithm in Action . . .

Lyndon words of length at most 4:

\[
\begin{align*}
\text{a} & \rightarrow \text{aaaa} & \rightarrow \text{aaab} \\
\text{ab} & \rightarrow \text{abab}
\end{align*}
\]
Duval’s Algorithm in Action . . .

Lyndon words of length at most 4:

\[
\begin{align*}
    a & \rightarrow aaaa & \rightarrow & \boxed{aaab} \\
    ab & \rightarrow abab & \rightarrow & abb
\end{align*}
\]
Duval’s Algorithm in Action ...

Lyndon words of length at most 4:

\[
\begin{align*}
    a & \rightarrow aaaa \\
    ab & \rightarrow abab \\
\end{align*}
\]

(repeat)
Duval’s Algorithm in Action . . .

Lyndon words of length at most 4:

- $a \rightarrow aaaa \rightarrow aaab$
- $ab \rightarrow abab \rightarrow abb$ (repeat)
- $aab$
Duval’s Algorithm in Action . . .

Lyndon words of length at most 4:

- \[ a \rightarrow aaaa \rightarrow aaab \]
- \[ ab \rightarrow abab \rightarrow abb \text{ (repeat)} \]
- \[ aab \rightarrow aaba \]
Duval’s Algorithm in Action . . .

Lyndon words of length at most 4:

\[
\begin{align*}
    a & \rightarrow aaaa \rightarrow aaab \\
    ab & \rightarrow abab \rightarrow abb \quad \text{(repeat)} \\
    aab & \rightarrow aaba \rightarrow aabb
\end{align*}
\]
Duval’s Algorithm in Action . . .

Lyndon words of length at most 4:

\[
\begin{align*}
    a & \rightarrow aaaa & \rightarrow & \text{aaab} \\
    ab & \rightarrow abab & \rightarrow & \text{abb} \text{ (repeat)} \\
    aab & \rightarrow aaba & \rightarrow & \text{aabb} \\
    abb & \\
\end{align*}
\]
Duval’s Algorithm in Action ... 

Lyndon words of length at most 4:

- $a \rightarrow aaaa \rightarrow aaab$
- $ab \rightarrow abab \rightarrow abb$ (repeat)
- $aab \rightarrow aaba \rightarrow aabb$
- $abb \rightarrow abba$
Duval’s Algorithm in Action . . .

Lyndon words of length at most 4:

\[
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   a & \rightarrow aaaa \rightarrow aaab \\
   ab & \rightarrow abab \rightarrow abb \text{ (repeat)} \\
   aab & \rightarrow aaba \rightarrow aabb \\
   abb & \rightarrow abba \rightarrow abbb
\end{align*}
\]
Duval’s Algorithm in Action . . .

Lyndon words of length at most 4:

- \[
\begin{align*}
\text{a} &\rightarrow aaaa \rightarrow \text{aaab} \\
\text{ab} &\rightarrow abab \rightarrow \text{abb} \text{ (repeat)} \\
\text{aab} &\rightarrow aaba \rightarrow \text{aabb} \\
\text{abb} &\rightarrow abba \rightarrow \text{abbb}
\end{align*}
\]

Updated List: \[ \mathcal{L} = \{a, b, ab, aab, abb, aaab, aabb, abbb, \ldots \} \]
Duval’s Algorithm in Action . . .

Lyndon words of length at most 4:

- $a$ → $aaaa$ → $aaab$
- $ab$ → $abab$ → $abb$ (repeat)
- $aab$ → $aaba$ → $aabb$
- $abb$ → $abba$ → $abbb$

Updated List: $\mathcal{L} = \{a, b, ab, aab, abb, aaab, aabb, abbb, \ldots\}$

And on it goes . . .
Remarks on Duval’s Algorithm

- The worst-case time to generate a successor of a Lyndon word \( w \) by Duval’s procedure is \( O(n) \).
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- The worst-case time to generate a successor of a Lyndon word $w$ by Duval’s procedure is $O(n)$.
- This can be improved to constant time if the generated words are stored in an array of length $n$ and the construction of $x$ from $w$ is performed by appending letters to $w$ instead of making a new copy of $w$. 
Remarks on Duval’s Algorithm

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- This can be improved to constant time if the generated words are stored in an array of length $n$ and the construction of $x$ from $w$ is performed by appending letters to $w$ instead of making a new copy of $w$. [Berstel-Pocchiola 1994]
Remarks on Duval’s Algorithm

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Duval (1983) also developed an algorithm for standard factorisation that runs in linear time and constant space.
Back to Factorisations of Lyndon Words

A famous theorem concerning Lyndon words asserts that every word $w$ can be uniquely factorised as a non-increasing product of Lyndon words . . .
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**Theorem**

Any word $w \in A^+$ may be uniquely written as a non-increasing product of Lyndon words, i.e.,

$$w = \ell_1 \ell_2 \cdots \ell_n$$

where the $\ell_i$ are Lyndon words such that $\ell_1 \succeq \ell_2 \succeq \cdots \succeq \ell_n$.
Back to Factorisations of Lyndon Words

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Theorem

Any word $w \in A^+$ may be uniquely written as a non-increasing product of Lyndon words, i.e.,

$$w = l_1 l_2 \cdots l_n$$

where the $l_i$ are Lyndon words such that $l_1 \succeq l_2 \succeq \cdots \succeq l_n$, called the Lyndon factorisation.

Example: $abaacaab$
Back to Factorisations of Lyndon Words

A famous theorem concerning Lyndon words asserts that every word $w$ can be uniquely factorised as a non-increasing product of Lyndon words . . .

**Theorem**

Any word $w \in A^+$ may be uniquely written as a non-increasing product of Lyndon words, i.e.,

$$w = \ell_1 \ell_2 \cdots \ell_n$$

where the $\ell_i$ are Lyndon words such that $\ell_1 \succeq \ell_2 \succeq \cdots \succeq \ell_n$, called the **Lyndon factorisation**.

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Back to Factorisations of Lyndon Words

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We note that the Lyndon factorisation of a word can be computed in linear time [Duval 1983].
Applications of Lyndon Factorisations

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- Interestingly, there is a relationship between Lyndon words, shift-register sequences, and de Bruijn words [Knuth 2005].
Some Applications of de Bruijn Sequences

- A de Bruijn sequence can be used to shorten a brute-force attack on a PIN-like code lock that does not have an “enter” key and accepts the last $n$ digits entered.
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Consider such a digital door lock with a 3-digit code.
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Consider such a digital door lock with a 3-digit code.

All possible solution codes are contained exactly once in $B(10, 3)$ — the de Bruijn sequence with alphabet $\{0, 1, \ldots, 9\}$ that contains each word of length 3 exactly once.
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- See [Wikipedia](https://en.wikipedia.org) for more information and other interesting applications.
Standard Factorisations

- Recall that if $w = uv$ is a Lyndon word with $v$ the longest proper Lyndon suffix of $w$, then $u$ is also a Lyndon word and $u \prec v$. 
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The left-standard and right-standard factorisations of $aabaacab$ are:

$$(aabaac)(ab) \quad \text{and} \quad (aab)(aacab).$$
Standard Factorisations . . .

The left-standard and right-standard factorisations of a Lyndon word sometimes coincide.
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$a, b, ab, ab, abb, aaab, abbb, aaaaab, aabab, ababb, abbbb, \ldots$

Do you notice any structural property that holds for all these words?
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All such words take the form \(aub\) where \(u\) is a palindrome.
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But not just any old palindrome . . .
Theorem (Melançon 1999, Berstel-de Luca 1997)

Suppose $w$ is finite word over $\{a, b\}$. Then $w$ is a Lyndon word with the property that its left and right standard factorisations coincide if and only if $w = aub$ (when $a \prec b$) or $w = bua$ (when $b \prec a$) where $u = Pal(v)$ for some word $v$ over $\{a, b\}$. 
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Suppose $w$ is a finite word over $\{a, b\}$. Then $w$ is a Lyndon word with the property that its left and right standard factorisations coincide if and only if $w = aub$ (when $a \prec b$) or $w = bua$ (when $b \prec a$) where $u = \text{Pal}(v)$ for some word $v$ over $\{a, b\}$.

- $\text{Pal}$ is the iterated palindromic closure operator [Justin 2005] defined as follows.
- For a given word $v$, let $v^+$ denote the unique shortest palindrome beginning with $v$, called the (right-)palindromic closure of $v$.
  For example:
  $$(glen)^+ = glenelg$$
  $$(race)^+ = racecar$$
- We define $\text{Pal}(\varepsilon) = \varepsilon$, and for any word $w$ and letter $x$,
  $$\text{Pal}(wx) = (\text{Pal}(w)x)^+.$$
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  $$\text{Pal}(abab) = \underline{aba}\underline{aba}\underline{baaba}$$
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For example:

$\text{Pal}(abab) = \underline{aba} \underline{bab} \underline{aba} \underline{ba} \underline{a} \underline{b} \underline{a} \underline{b} \underline{a}$

$\rightarrow$ \textit{coincidental Lyndon word}: \text{aPal}(abab)b = \underline{a} \underline{a} \underline{b} \underline{a} \underline{b} \underline{a} \underline{b} \underline{a} \cdot \underline{a} \underline{a} \underline{b} \underline{a} \underline{b}
The previous theorem says that the “coincidental Lyndon words” over \{a, b\} with \(a \prec b\) are precisely the words \(a \text{Pal}(v)b\) where \(v \in \{a, b\}^*\).
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These words are known as (lower) \textbf{Christoffel words} (named after E. Christoffel 1800’s) – they can be constructed geometrically by the coding the horizontal and vertical steps of certain \textbf{lattice paths} . . .
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- Consider a line (call it \( \ell \)) of the form:

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y = \frac{p}{q}x
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where \( p, q \) are positive integers with \( \gcd(p, q) = 1 \).
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- Consider a line (call it \(\ell\)) of the form:

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where \(p, q\) are positive integers with \(\gcd(p, q) = 1\).

- Let \(\mathcal{P}\) denote the path along the integer lattice below the line \(\ell\) that starts at the point \((0, 0)\) and ends at the point \((q, p)\) with the property that the region in the plane enclosed by \(\mathcal{P}\) and \(\ell\) contains no other points in \(\mathbb{Z} \times \mathbb{Z}\) besides those of the path \(\mathcal{P}\).
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- The so-called lower Christoffel word of slope \( p/q \), denoted by \( L(p, q) \), is obtained by coding the steps of the path \( \mathcal{P} \).
The previous theorem says that the “coincidental Lyndon words” over \{a, b\} with \(a < b\) are precisely the words \(a Pal(v)b\) where \(v \in \{a, b\}\.*

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  - A vertical step is denoted by the letter \( b \).
Christoffel Construction by Lattice Paths

Lower Christoffel word of slope $\frac{3}{5}$
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$L(3, 5) = a$
Christoffel Construction by Lattice Paths

Lower Christoffel word of slope $\frac{3}{5}$

$L(3, 5) = aa$
Christoffel Construction by Lattice Paths

Lower Christoffel word of slope $\frac{3}{5}$

$L(3, 5) = aab$
Christoffel Construction by Lattice Paths

Lower Christoffel word of slope $\frac{3}{5}$

$L(3, 5) = aaba$
Christoffel Construction by Lattice Paths

Lower Christoffel word of slope $\frac{3}{5}$

$L(3, 5) = aabaa$
Christoffel Construction by Lattice Paths

Lower Christoffel word of slope $\frac{3}{5}$

$L(3, 5) = aabaab$
Christoffel Construction by Lattice Paths

Lower Christoffel word of slope \( \frac{3}{5} \)

\[ L(3, 5) = aabaaba \]
Christoffel Construction by Lattice Paths

Lower Christoffel word of slope $\frac{3}{5}$

$L(3, 5) = aabaabab$
Christoffel Words

Lower & Upper Christoffel words of slope $\frac{3}{5}$

$L(3, 5) = aabaabab$  
$U(3, 5) = babaabaa$
Remarks

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Thank You!